ON THE DEAD END DEPTH OF THOMPSON’S GROUP $F$

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Abstract. Thompson’s group $F$ was introduced by Richard Thompson in the 1960’s and has since found applications in many areas of mathematics including algebra, logic and topology. We focus on the dead end depth of $F$, which is the minimal integer $N$ such that for any group element, $g$, there is guaranteed to exist a path of length at most $N$ in the Cayley graph of $F$ leading from $g$ to a point farther from the identity than $g$ is. By viewing $F$ as a diagram group, we improve the greatest known lower bound for the dead end depth of $F$ with respect to the standard consecutive generating sets.

1. Introduction

The dead end depth of a group with respect to a finite generating set $S$ is the minimal integer $N$ such that such that for any group element, $g$, the distance from $g$ to the complement of the ball of radius $d(e,g)$ centered at the identity, $e$, is at most $N$. Among other places, dead ends have found application in the proof in [10] demonstrating a random walk that is biased towards the identity on the lamplighter group but that escapes from the identity faster than a simple random walk. Dead ends also played a role in Bogopol’skiǐ’s result that infinite commensurable hyperbolic groups are bi-Lipschitz equivalent [1].

A common theme in Geometric Group Theory is to classify the generating sets with respect to which a certain group or class of groups possess a given property. For dead end depth, few definitive results of this kind are known. Bogopol’skiǐ [1] proved that the depth of a hyperbolic group with respect to any finite generating set is finite. Lenhart [9] has shown abelian groups and groups with more than one end in the sense of Hopf and Freudenthal have bounded dead end depth. In the other direction, Cleary and Riley [3, 4] construct a finitely presented group with unbounded dead end depth with respect to a particular generating set. And, Riley and Warshall [11] have shown that having unbounded dead end depth is not a group invariant, that is, that it depends on the choice of generating set.

For Thompson’s group $F$, the exact dead end depth is known only for the standard generating set, $\{x_0, x_1\}$, in which case the depth is known to be 3 [5]. For the larger standard consecutive generating sets, $X_n = \{x_0, x_1, \ldots, x_n\}$, the depth is known only to be bounded between $\frac{4}{7}$ and $4n - 2$ [8]. In this paper we prove the following theorem raising the known lower bound for the dead end depth of $F$ with respect to $X_n$ to $2n - 7$.

Main Theorem. For $n \geq 4$, The dead end depth of Thompson’s group $F$ with respect to the generating set $X_n = \{x_0, x_1, \ldots, x_n\}$ is at least $2n - 7$.

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This paper is organized as follows: In the next section we formally define Thompson’s Group $F$ and review some necessary background information and results about $F$. In that section, we also formally define the concept of dead end depth. Section 3 is devoted to developing the methods and tools necessary to prove our main theorem and to proving the theorem assuming the lemmas of that section. The fourth section holds the proofs to the lemmas introduced in the third section, which are somewhat technical.

2. Background

2.1. Introduction to Thompson’s Group $F$. We now give a brief introduction to Thompson’s Group $F$. For a more detailed explanation, we refer the reader to [2]. As a set, Thompson’s Group $F$ is the set of orientation preserving piecewise linear homeomorphisms, $f$, from the unit interval $I$ to itself such that:

1. There are only finitely many points of non-differentiability of $f$,
2. Each point of non-differentiability of $f$ occurs at a dyadic rational number,
3. Every slope of $f$ is a power of 2.

The group operation of $F$ is composition of functions.

The group $F$ is usually studied by the following infinite and finite presentations.

\begin{align*}
F & = \langle x_k, k \geq 0 \mid x_i^{-1}x_jx_i = x_{j+1} \text{ if } i < j \rangle \\
F & = \langle x_0, x_1 \mid [x_0x_1^{-1}, x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2] \rangle
\end{align*}

The elements $x_0$ and $x_1$ are the functions given as follows with their graphs.

\[x_0(t) = \begin{cases} 
\frac{1}{2}t, & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{2}t - \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4} \\
2t - 1, & \frac{3}{4} \leq t \leq 1
\end{cases}\]

\[x_1(t) = \begin{cases} 
t, & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{2}t + \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4} \\
t - \frac{1}{4}, & \frac{3}{4} \leq t \leq \frac{7}{8} \\
2t - 1, & \frac{7}{8} \leq t \leq 1
\end{cases}\]

Note that the graph of $x_1$ consists of the graph of the identity on the first half of $I$ together with a copy of the graph of $x_0$ that has been “scaled down” by a factor one half and placed in the upper right hand corner. This means that one can think of $x_1$ as “acting as $x_0$” on the subinterval $[\frac{1}{2}, 1]$ of $I$ and as the identity elsewhere.
For $n \geq 1$, the generating sets $X_n := \{x_0, x_1, x_2, \ldots, x_n\}$ are referred to as the standard consecutive generating sets and $X_\infty = \{x_0, x_1, x_2, \ldots\}$ is referred to as the standard infinite generating set for $F$. With respect to $X_\infty$, each element $g$ of $F$ can be expressed uniquely in normal form,

$$g = x^{r_1}_{i_1} x^{r_2}_{i_2} \cdots x^{r_k}_{i_k} x^{-s_1}_{j_1} \cdots x^{-s_k}_{j_k} x^{-s_1}_{j_1}$$

with $s_i, r_i > 0$, $i_1 < i_2 < \cdots < i_k$, and $j_1 < j_2 < \cdots < j_k$ and such that if both $x_i$ and $x_i^{-1}$ appear in the expression then so does $x_j$ or $x_j^{-1}$ for some $j > i$ (see [2]). The normal form is an important tool in the study of the combinatorial properties of $F$. One important property of the normal form is that it is a minimal length expression for $g$ with respect to $X_\infty$. The normal form also provides a canonical way to decompose an element $g$ as the product of a “positive” element and a “negative” element. Explicitly, if the normal form representation of $g$ is $g = x^{r_1}_{i_1} x^{r_2}_{i_2} \cdots x^{r_k}_{i_k} x^{-s_1}_{j_1} \cdots x^{-s_k}_{j_k} x^{-s_1}_{j_1}$, then we write $g_+ = x^{r_1}_{i_1} x^{r_2}_{i_2} \cdots x^{r_k}_{i_k} x^{-s_1}_{j_1} x^{-s_1}_{j_1}$ and $g_- = x^{-s_2}_{j_2} x^{-s_1}_{j_1}$, and call $g_+$ the “positive” portion of $g$ and $g_-$ the “negative” portion of $g$.

2.2. Diagram representations of elements of $F$. Because it is difficult to understand the effect of composing two elements of $F$ by studying their piecewise formulas, elements of $F$ are frequently represented combinatorially. One combinatorial representation for elements of $F$ is the infinite diagrams described in [6] and [7], whose definition we now review.

**Definition 2.1.** An infinite diagram is a directed planar graph $D$ with infinite vertex set $\{v_0, v_1, v_2, \ldots\}$ together with an embedding of $D$ in the plane in such a way that:

- For each $i$, there is an edge directed from $v_i$ to $v_{i+1}$.
- The $v_i$ are discrete points of the $x$-axis and the edge from $v_i$ to $v_{i+1}$ is smoothly embedded in the $x$-axis. Such edges are called central edges.
- All other edges are smoothly embedded in the plane either entirely above or entirely below the $x$-axis and are directed left to right.
- All finite regions have three edges. Such regions are called cells.
- There are only a finite number of non-central edges in $D$.

Since a diagram has only finitely many non-central edges, there is a rightmost vertex $v_k$ incident to a non-central edge. We call the subgraph of $D$ spanned by vertices $\{v_1, v_2, \ldots, v_k\}$ the essential portion of $D$. Generally, we omit the infinite right tail of a diagram from figures and display only the essential portion of the diagram in questions. Figure 1 shows an example of the essential portion of a diagram, with directions on the edges omitted. We require some specialized graph-theoretical terminology specific to diagrams, which we now describe. The initial and terminal vertices of edge $e$ are denoted by $i(e)$ and $t(e)$ respectively. An upper (respectively lower) edge is an edge which lies completely above (respectively below) the central line of a diagram. The lower edge $e_1$ is below the lower or central edge $e_2$ if $i(e_1)$ is equal to or left of $i(e_2)$ and $t(e_1)$ is equal to or to the right of $t(e_2)$ on the $x$-axis. Similarly, the upper edge $e_1$ is above the upper or central edge $e_2$ if the same condition on the endpoints of $e_1$ and $e_2$ holds. A lower or central edge $e$ is exposed from the bottom if there is no edge below $e$ and the upper or central edge $e$ is exposed from the top if there is no edge above $e$. The bottom path of $D$ is the path consisting of the set of central or lower edges that are exposed from the
bottom and the upper path is the path consisting of the set of upper or central edges exposed from the top. Vertices on the bottom (respectively top) path are exposed from the bottom (respectively top). A cell $C$ lying above the $x$-axis is an exposed upper cell if two of its edges are central edges that are exposed from the bottom.

Vertices $v_1, v_2$ and $v_3$ are the vertices of an exposed cell in Figure 1.

![Figure 1. Essential portion of a diagram with upper exposed cell](image)

Note that the two central edges of an exposed upper cell are necessarily embedded adjacent to each other on the $x$-axis. Finally a vertex $v$ is said to be covered by the bottom edge $e$ if $i(e)$ is to the left of $v$ and $t(e)$ is to the right of $v$.

In order to establish a one-to-one correspondence between infinite diagrams and elements of $F$, one must restrict attention to the reduced diagrams, which we now define.

**Definition 2.2.** A dipole in the infinite diagram $D$ is a subgraph $D_0$ of $D$ consisting of three vertices, $v_i, v_{i+1}$ and $v_{i+2}$ together with the central edges between them, an upper edge from $v_i$ to $v_{i+2}$ and also a lower edge from $v_i$ to $v_{i+2}$ as shown in Figure 2.

![Figure 2. A dipole](image)

**Definition 2.3.** The diagram $D$ is reduced if it contains no subgraph that is a dipole.

Two infinite diagrams $D_1$ and $D_2$ are isomorphic if there is an orientation-preserving homeomorphism of $\mathbb{R}^2$ to itself that restricts to a direction-preserving graph isomorphism between $D_1$ and $D_2$ taking central edges to central edges, upper edges to upper edges and lower edges to lower edges. Since it is the set of isomorphism classes of reduced diagrams that is in one-to-one correspondence with elements of $F$, it is convenient to abuse notation and refer to the “diagram $D$” when one really means the “isomorphism class of diagram $D$”. For the remainder of the paper, we adopt the convention of simply referring to “diagrams” instead of “isomorphism classes of diagrams”. Additionally, for the remainder of the paper, all diagrams under consideration will be infinite, reduced diagrams, but we omit
the repetition of the words “infinite” and “reduced”. Henceforth, all diagrams are assumed to be infinite reduced diagrams.

The correspondence between the set of reduced diagrams and $F$ is most easily defined by giving each vertex of a diagram $D$ an “upper” and “lower” label, defined as follows. Let $w_0, w_1, w_2, \ldots$ be the vertices of the bottom path in order from left to right and let $u_0, u_1, u_2, \ldots$ be the vertices of the top path in order from left to right. Vertex $w_i$ has bottom label $\frac{2^i - 1}{2^i}$ and vertex $u_i$ has top label $\frac{2^{i+1} - 1}{2^{i+1}}$. The bottom labels of the remaining vertices without bottom labels are defined by the property that if $C$ is a lower cell with vertices $v_i, v_j, v_k$ with $i < j < k$ then the bottom label of $v_j$ is the average of the bottom labels of $v_i$ and $v_k$. The top labels of the remaining vertices without top labels are defined similarly. Figure 3 illustrates the labeling of the essential portion of a diagram.

![Figure 3. A labeled diagram](image)

We are now in a position to define the correspondence between the set of (isomorphism classes of reduced, infinite) diagrams and $F$. For diagram $D$, denote by $\text{top}(v)$ and $\text{bot}(v)$ the top and bottom labels of vertex $v$. The orientation preserving piecewise linear homeomorphism of $I$ with itself that corresponds to $D$ is the function $f_D$ defined by:

1. $f_D(\text{bot}(v)) = \text{top}(v)$ for every vertex $v$ of $D$
2. $f_D$ is linear on the complement of the set of bottom labels of $D$
3. $f_D$ is an orientation preserving piecewise linear homeomorphism from $I$ to itself.

For example, the function given by the diagram in Figure 3 maps the interval $[0, \frac{1}{4}]$ linearly to $[0, \frac{1}{2}]$, the interval $[\frac{1}{4}, \frac{3}{8}]$ linearly to the interval $[\frac{1}{2}, \frac{3}{4}]$, and so on. The (isomorphism class of reduced) diagram corresponding to an element of $F$ in this way is unique, and we denote the diagram corresponding to $g$ by $D(g)$.

The diagram representations of the elements of $x_0$ and $x_1$ are shown in Figure 4. As usual, we omit the infinite right tail of central vertices and show only the essential portion of the diagrams.

![Figure 4. Diagrams for $x_0$ and $x_1$](image)
In general, $x_i$ corresponds to the diagram with a single upper edge from vertex $v_i$ to $v_{i+2}$, and the diagram for $x_i^{-1}$ is constructed by reflecting the diagram for $x_i$ about the $x$-axis.

An important relationship between the diagram for an element and its normal form is the fact that if we consider $\alpha_+$ and $\alpha_-$ as elements of $F$ in their own rights, then the diagram for $\alpha_+$ is the diagram consisting of the upper and central edges from the diagram of $\alpha$ and the diagram of $\alpha_-$ is the diagram consisting of central and lower edges of the diagram of $\alpha$. Thus, the number of upper (respectively lower) edges in the diagram for $g_n$ is equal to the number of positive (respectively negative) generators (counted with multiplicity given by their exponents) occurring in the normal form of $g$.

For the purposes of this paper, we need only to multiply general elements of $F$ by single generators in $X_n$ at a time. In order to see the effect of multiplication by a generator, consider an element $g$ in $F$ with diagram $D(g)$. Label by $w_0, w_1, w_2, \ldots$ the vertices along the bottom path of $g$ starting with the vertex labeled 0 on top and bottom. Now consider a generator $x_i \in X_n$.

To produce the diagram for $g \cdot x_i$, proceed as follows:

- If there exists one or more lower edges directed out of $w_i$, remove the bottom-most edge.
- If there is no lower edge directed out of $w_i$, alter $D(g)$ by adding an upper edge out of $w_i$, terminating at $w_{i+1}$ below all upper edges and subdividing the central edge out of $w_i$ into two central edges, one originating at $w_i$ terminating at a new vertex $w$ and one originating at the new vertex $w$ and terminating at $w_{i+1}$.

Figure 5 illustrates the result of the modification in the second case with $x_3$.

To get the diagram for $g \cdot x_i^{-1}$, modify the diagram for $g$ as follows:

- If $w_i$ is the first vertex of an exposed upper cell, eliminate the three edges of that cell and the central vertex it contains, and add a new central edge connecting $w_i$ and $w_{i+2}$.
- If $w_i$ is not the initial vertex of an exposed cell, add a bottom-most lower edge initiating at $w_i$ and terminating at $w_{i+2}$.

2.3. Dead Ends. The length of an element $g \neq id$ of a group $G$ with respect to a generating set $S$ is the minimal length of a product of elements from $S \cup S^{-1}$ that is equal to $g$. Geometrically, the length of $g$ is the distance from the identity element to $g$ in the Cayley Graph of $G$ with respect to the generating set $S$. We denote by $l_S(g)$ the length of $g$ with respect to the generating set $S$. If a there is a fixed generating set $S$ under consideration we occasionally omit the mention of the generating set and write simply $l(g)$ for $l_S(g)$. We now define dead ends.
Definition 2.4. Let $G$ be an infinite group, and let $S$ be a finite generating set for $G$. The element $g \in G$ is a dead end element with respect to $S$ if:

$$l(gs) \leq l(g) \forall s \in (S \cup S^{-1}).$$

Definition 2.5. Let $G$ be a group, and let $S$ be a finite generating set for $G$. The dead end depth of element $g$ in $G$ is given by,

$$\text{depth}(g) := \min \{l(h) \mid l(gh) > l(g)\}.$$

We remark that with this definition, an element that is not a dead end has dead end depth 1, and a dead end has depth at least 2. This is consistent with the convention in, for example, [3] and [5] but is inconsistent with the convention in [8], where the dead end depth of an element is one less than the depth given here.

Definition 2.6. The dead end depth of finitely generated group $G$ with respect to the finite generating set $S$ is the maximum $N$ that is the dead end depth of an element in $G$ if such an $N$ exists, and is infinity otherwise.

2.4. Calculating Length in $F$. Dead ends are known to exist in Thompson’s Group $F$ with respect to any consecutive generating set $X_n$ [5, 8]. As mentioned in the introduction, the exact dead end depth of $F$ is known only with respect to $X_1$, and for $n \geq 3$ the dead end depth is known only to be bounded between $\frac{n}{2}$ and $4n - 2$. Using infinite diagrams to analyze the effect on length of multiplication by a generator, we improve the lower bound by proving the following Theorem.

Theorem 2.1. For $n \geq 4$, The dead end depth of Thompson’s group $F$ with respect to the generating set $X_n = \{x_0, x_1, \ldots, x_n\}$ is at least $2n - 7$.

Our main tool is the formula in [8] for determining the length of an element of $F$ with respect to $X_n$, which reads as follows.

Theorem 2.2 ([8] Theorem 3.3). For every $g \in F$, the word length of $g$ with respect to the generating set $X_n$ is given by the formula,

$$l_n(g) = l_\infty(g) + 2P_n(g)$$

where $l_\infty(g)$ is the total number of upper and lower edges of the diagram for $g$, and $P_n(g)$ is the penalty weight of $g$.

As mentioned during the discussion of diagrams, the first component of the word length formula, $l_\infty(g)$, is the word length of the element $g$ with respect to the infinite generating set $X_\infty$, which is also the number of generators appearing in the normal form representation of $g$, and the number of non-central edges in $D(g)$. The second component, $P_n(g)$, is the so-called penalty weight of $g$, which we now define.

A vertex in $D(g)$ is of penalty type if it is the initial vertex of a non-central edge, or it is a separating vertex whose removal separates $D(g)$ into two components with the right component containing a non-central edge. Such a vertex will be called an “essential cut” vertex. Even though in the strict graph-theoretical sense of the term, any vertex of $D(g)$ to the right of the last non-central edge is a cut vertex, these are not “essential” cut vertices in our definition. Recall that $D(g)$ is a directed graph, with edges directed from left to right. A penalty tree in the diagram $D(g)$ is a directed subtree of $D(g)$ rooted at the vertex $v_0$ that contains a directed path from $v_0$ to each penalty type vertex of $D(g)$. We note that a penalty tree may contain non-penalty type vertices.
The penalty weight, $P_n(T)$ of penalty tree $T$ with respect to $X_n$ is the number of vertices $w$ (penalty type or not) in $T$ satisfying,
1. The distance from $v_0$ to $w$ through $T$ is greater than or equal to 2,
2. There is a directed path through $T$ of length at least $n-1$ from $w$ to a leaf $\ell$ of $T$.

The penalty weight, $P_n(g)$, of $g$ with respect to $X_n$ is the minimal penalty weight of all penalty trees in $D(g)$. That is, $P_n(g) = \min\{P_n(T) \mid T \text{ is a penalty tree of } g\}$

For example consider the element, $g$, whose diagram is shown in Figure 6. Now, $l_\infty(g) = 15$ and the penalty tree indicated by the dashed edges has penalty weight 3. Thus, our length formula gives us that $l_3(g)$ is at most 21. It is not hard to show that the given $T$ is in fact of minimal weight among all penalty trees for $g$, so $l_3(g) = 21$.

3. Increased lower bound for dead end depth

In this section we fix an integer $n \geq 4$ and develop the tools used in the proof of our main theorem. The proofs are somewhat technical and we postpone the most technical ones until Section 4.

3.1. The Element $g_n$. The proof that the dead end depth of $X$ with respect $X_n$ is at least $2n-7$ consists simply of exhibiting an element $g_n$, which we formally define below, whose dead end depth is shown to be at least $2n-7$. The diagram of $g_n$ is constructed by repeating a “principal section” $7n$ times. This section is defined in such a way as to prevent multiplication by any element of $F$ of length less than or equal to $2n-8$ with respect to $X_n$ from altering the top half of the diagram for $g_n$.

The principal section of $g_n$ is the diagram, denoted by $S_n$, that has $2^{2n-4} + 1$ vertices, $\{s_0, s_1, \ldots, s_{2^{2n-4}}\}$ with central edges from $s_i$ to $s_{i+1}$ for each $i$. Additionally, there is an upper edge from $s_1$ to $s_3$, and for $i \geq 3$ there is an upper edge from $s_0$ to $s_i$. The lower edges are constructed recursively as follows:
1. There is a lower edge from $s_0$ to $s_{2^{2n-4}}$, the “level $2n-4$” lower edge,
2. For every level $k \geq 2$ lower edge from vertex $s_i$ to $s_j$, there is a level $k-1$ lower edge from $s_{i+1}$ to $s_{i+2}$ and a level $k-1$ lower edge from $s_{i+1}$ to $s_j$,
3. A total of $2n-4$ levels of lower edges are added.

The diagram for the element $g_n$ is constructed by gluing $6n$ copies of the diagram $S_n$ together end to end, as shown in Figure 7. Note that $g_n$ has $2n-4$ levels of lower edges, the first three vertices of each principal section are penalty type vertices, and after that every other vertex of each principal section is a penalty type vertex.

In order to show that the dead end depth of $G_n$ is at least $2n-7$, it is not necessary to establish the exact length of $g_n$ with respect to $X_n$, but it is necessary
to find a minimal weight penalty tree. It is not difficult to show that a minimal penalty tree \( T_n \) may be constructed without using any lower edge of level greater than 1 as follows. The vertex set of \( T_n \) is the set of penalty type vertices of \( g_n \). The first central edge in each principal section of \( g_n \) belongs to \( T_n \). The leftmost level 1 lower edge of each principal section belongs to \( T_n \). And, for any penalty type vertex \( v \) of \( g_n \) that is not the first, second or third vertex of a section, the unique upper edge terminating at \( v \) belongs to \( T_n \).

Since it contains all penalty type vertices of \( g_n \), the subtree \( T_n \) is a penalty tree for \( g_n \). To get a sense of which vertices of \( T_n \) are weighted vertices, note that \( T \) consists of one long path consisting of the uppermost edges in the essential portion of the diagram of \( g_n \) together with many single edges attached to the essential cut vertices. Therefore, the only weighted vertices in \( T_n \) are those essential cut vertices \( w \) that are a distance at least \( n - 2 \) from the last essential cut vertex along the upper path of \( g_n \). The distance in the previous sentence is \( n - 2 \) because the edges in \( T_n \) originating at this vertex make \( w \) at distance at least \( n - 1 \) from a leaf.

Because every penalty tree of \( g \) must contain all of the essential cut vertices of \( g \), it is not difficult to show that \( T_n \) is actually a minimal penalty tree for \( g \). Moreover, the only non-weighted vertices in \( T_n \) that are on the upper path of \( g_n \) are \( v_0 \), the first vertex of the second principal section and those that are at distance greater than \( 4n \) from \( v_0 \), though not all such vertices are actually weighted. Additionally, multiplication of \( g_n \) by an element \( \alpha \in F \) of length less than \( 2n - 7 \) does not modify any part of the diagram of \( g_n \) farther to the right than the \( 4n^{th} \) principal section. For these two reasons, we call the portion of \( D(g_n) \) consisting of the first \( 4n \) principal sections the active portion of \( D(g_n) \).

This allows us to prove,

**Lemma 1.** Let \( \alpha \) be an element of \( F \) with \( l_n(\alpha) \leq 2n - 8 \) such that the normal form of \( \alpha \) has more positive generators than negative, then \( l_n(g_n\alpha) \leq l_n(g_n) \).

**Proof.** Let \( \alpha = g = x_{i_1}^{r_1}x_{i_2}^{r_2} \cdots x_{i_k}^{r_k}x_{j_1}^{-s_1} \cdots x_{j_2}^{-s_2}x_{j_3}^{-s_3} \) be the normal form for \( \alpha \). Since the normal form of \( \alpha \) contains more positive generators than negative generators, \( r_1 + r_2 + \cdots + r_k > s_1 + s_2 + \cdots + s_3 \). Note that \( \alpha \) has at most \( 2n - 8 \) upper edges total since \( l_\infty(\alpha) \leq l_n(\alpha) \leq 2n - 8 \). We consider the process of constructing the diagram for \( g_n\alpha \) from the diagram for \( g_n \) by analyzing the effect of multiplying by one generator from the normal form of \( \alpha \) at a time starting with \( x_{i_1} \). Since \( g_n \) has \( 2n - 4 \) levels of lower edges, the \( m^{th} \) positive generator from the normal form of \( \alpha \) is guaranteed to have the effect of removing one lower edge of level at least 2

*Figure 7. Sample \( g_n \) Diagram (edges of level less than \( 2n - 2 \) not shown)*
from the diagram resulting from multiplying by the first $m-1$ positive generators. Thus, $l_\infty(g_nx_1^1x_2^2 \cdots x_k^k) = l_\infty(g_n) - (r_1 + r_2 + \cdots + r_k)$.

Also, since the removal of at most $2n - 8$ lower edges from $D(g)$ cannot expose an upper cell, the $m^{th}$ negative generator from the normal form has the effect of adding one lower edge to the diagram resulting from multiplying by all of the positive generators and the first $m-1$ negative generators. Since $r_1 + r_2 + \cdots + r_k > s_1 + s_2 + \cdots + s_l$, we find that $l_\infty(g_n,\alpha) \leq l_\infty(g_n)$.

Now, since $\alpha$ has at most $2n - 8$ upper edges, the diagram for $g_n,\alpha$ differs from the diagram for $g$ only in lower edges of level greater than 1. Thus, the tree $T'_n$ drawn in the diagram for $g_n,\alpha$ exactly as the tree $T_n$ in the diagram for $g_n$ is a valid penalty tree for $g_n,\alpha$. Thus, $p_n(g_n,\alpha) \leq p_n(g_n)$. Therefore, $l_n(g_n,\alpha) = l_\infty(g_n,\alpha) + 2p_n(g_n,\alpha) \leq l_\infty(g_n) + 2p_n(g_n) = l_n(g_n)$, as required.

3.2. Informal Strategy to determine the depth of $g_n$. Here we heuristically describe the strategy used to determine the dead end depth of $g_n$. Informally, we view the situation as follows. No multiplication of $g_n$ by an element $\alpha$ with $l_n(\alpha) \leq 2n - 8$ can modify the active section of $D(g_n)$ since such the diagrams of such elements contain at most $2n - 8$ edges, all of which originate at vertices closer than $4n$ to $v_0$. Therefore, multiplication of $g_n$ such an $\alpha$ cannot create a penalty type vertex. For the same reason, none can eliminate an edge of $T_n$. This means that the only way to increase the length of $g_n$ by multiplying it by an element of length less than $2n - 8$ would be to find a way to add a set of edges whose net effect does not reduce the penalty weight. The general strategy for doing so is displayed in Figure 8. The ovals represent the principal sections of $G_n$ and the additional lines are the added edges.

Recall that all essential cut vertices must belong to every penalty tree $T$ of $g_n$. Now, if an essential cut vertex of $g_n$ is covered below by an edge added by multiplying $g_n$ by an element of $X_n \cup X_n^{-1}$ then the penalty tree may be modified to eliminate it as a weighted vertex by using the new edge instead of the upper edge terminating at the new edge’s endpoint. This modification has the effect of removing the covered vertex from the single long path in $T_n$ and preventing it from being at distance $n - 1$ from a leaf. This holds true not only for multiplication of $g_n$ by generators, but for the multiplication of $g_n,\alpha$ by generators provided that $\alpha$ added no more than $n - 2$ edges to $D(g)$. After $n - 2$ edges are added, length can start to increase again. Since the first $n - 2$ edges reduced the penalty weight and thus the actual length, a total of $2n - 3$ edges would be required to increase length beyond that of $g_n$. Thus, in order for $l_n(g_n,\alpha)$ to be greater than $l_n(g_n)$, $l_\infty(\alpha)$ must be at least $2n - 3$. But not every set of $2n - 3$ additional edges results in the
increase of \(l_n(g_n)\). The edges must be placed in such a way that the last \(n - 2\) of them actually do make length start to go back up after the first \(n - 2\) decreased length. In order to do this, they all must terminate sufficiently far away from \(v_0\). Such a concern seems to force the penalty weight of such an element to be at least \(n - 3\) for a total length of \(4n - 9\) with respect to \(X_n\). This heuristic argument leads to the following.

**Conjecture 3.1:** For \(n \geq 2\), the depth of Thompson’s Group \(F\) with respect to a generating set \(X_n = \{x_0, x_1, \ldots, x_n\}\) is greater than or equal to \(4n - 9\).

### 3.3. Penalty trees in \(g_n\alpha\)

In order to analyze the effect on length of multiplying \(g_n\) by an element \(\alpha\) of length at most \(2n - 8\), we must construct penalty trees in \(D(g_n, \alpha)\) that, if not minimal, are at least close enough to minimal to allow us to show that length does not increase. We may have to apply this construction to multiplication by elements that have length greater than \(2n - 8\) with respect to \(X_n\), but those elements will always have length at most \(2n - 8\) with respect to \(X_\infty\), and moreover multiplying \(g_n\) by them will have the effect of modifying only the active portion of \(D(g_n)\).

So, consider an element \(\alpha \in F\) of length at most \(2n - 8\) with respect to \(X_\infty\) such that multiplication of \(g_n\) by \(\alpha\) involves changing only the active portion of \(D(g_n)\). Let \(\alpha = \alpha_+\alpha_-\) with \(\alpha_+\) and \(\alpha_-\) the positive and negative portions respectively of the normal form of \(g_n\). Suppose that \(\alpha_+\) contains \(k\) generators and \(\alpha_-\) contains \(l\) generators. Since \(l_\infty(\alpha) \leq 2n - 8\), the effect on the diagram of multiplication of \(g_n\) by \(\alpha\) is the removal of \(k\) lower edges followed by the addition of \(l\) new lower edges with \(k, l \leq 2n - 8\). We now construct a penalty tree for \(g_n\alpha\). We remark that the set of penalty type vertices of \(g_n\alpha\) is exactly the set of penalty vertices of \(g_n\).

**Definition 3.1.** For element \(\alpha \in F\) as above, construct a penalty tree \(\tilde{T}\) in \(g_n\alpha\) by adding edges together with their endpoints as follows.

1. \(\tilde{T}\) contains all edges in the bottom path of \(g_n\alpha\).
2. Recursively add additional edges to \(\tilde{T}\) by following the following steps.
   (a) Scan the diagram right to left starting at the rightmost vertex incident to a non-central edge until an essential cut vertex of \(g_n\), say \(w\), is encountered that is not already contained in \(\tilde{T}\).
   (b) Add to \(\tilde{T}\) the lower edge \(e_1\) originating farthest left and terminating at \(w\). From the initial vertex of \(e_1\) add the lower edge \(e_2\) terminating at \(i(e_1)\) and originating the farthest left. Continue adding edges originating farthest left until a vertex already in \(\tilde{T}\) is reached.
   (c) Repeat (a) and (b) until every essential cut vertex of \(g_n\) is contained in \(\tilde{T}\).
3. For any penalty type vertex \(v\) not yet in \(\tilde{T}\), add to \(\tilde{T}\) the non-lower (i.e upper or central) edge which terminates at \(v\) and originates at an essential cut vertex of \(g_n\), if such an edge exists.
4. Any penalty type vertex \(v\) that is not yet in \(\tilde{T}\) is the third vertex in a principal section of \(\tilde{T}\). For such a vertex \(v\), add the unique lower edge terminating at \(v\).

We postpone the technical proofs of the following four lemmas to Section 4.

**Lemma 2.** Let \(\alpha \in F\) have length at most \(2n - 8\) with respect to \(X_\infty\). Suppose that the normal form of \(\alpha\) contains the same number, \(k\), of positive and negative
generators and that multiplication of \( g_n \) by \( \alpha \) involves modifying only the active portion of \( D(g_n) \). If vertex \( w \) of \( g_n\alpha \) that is an essential cut vertex of \( g_n \) is not contained in the bottom path of \( D(g_n\alpha) \), then for the first vertex \( b \) to the left of \( w \) through \( \bar{T} \) that is also on the bottom path of \( D(g_n\alpha) \), we have \( d_{\bar{T}}(w,b) \leq k \).

**Lemma 3.** Let \( \alpha \in F \) have length at most \( 2n - 8 \) with respect to \( X_\infty \), and suppose that multiplication of \( g_n \) by \( \alpha \) involves changing only the active portion of \( D(g_n) \). If the normal form of \( \alpha \) contains the same number, say \( k \), of positive as negative generators then \( P_n(\bar{T}) \leq P_n(g_n) \).

We remark that, though we do not need it for the proofs that follow, Lemma 3 implies that, under its hypotheses, \( l_n(g_n\alpha) \leq l_n(g_n) \).

To determine the effect on length of multiplying by an element \( \alpha \) of length at most \( 2n - 4 \) whose normal form does not contain an equal number of positive and negative generators, we must carefully analyze how much more difficult it is to reach penalty type vertices through \( \bar{T} \) than it is through \( T_n \). For this, we require the following definition.

**Definition 3.2.** A vertex \( v \) is said to be \( d \) levels deep if there are exactly \( d \) bottom edges covering \( v \).

**Lemma 4.** Let \( \alpha \) be as in Lemma 2, and let \( w \) and \( b \) be vertices of \( g_n\alpha \) such that:

- \( w \) is an essential cut vertex of \( g_n \),
- \( w \) is covered by a bottom edge of \( g_n\alpha \),
- \( b \) is the first vertex left of \( w \) along \( \bar{T} \) that is on the bottom path of \( g_n\alpha \), and
- \( w \) is at most \( j \) levels deep.

Then using the penalty tree \( \bar{T} \) from Definition 3.1, \( d_{\bar{T}}(b,w) \leq j \).

**Lemma 5.** Let \( \alpha \in F \) with \( l_\infty(\alpha) \leq 2n - 8 \) and let \( \beta^{-1} \in X_n \). Suppose further that multiplication of \( g_n \) by \( \alpha \beta \) involves changing only the active portion of \( D(g_n) \). Denote by \( \bar{T}_1 \) the penalty tree in \( g_n\alpha \) constructed in Definition 3.1 and \( \bar{T}_2 \) the tree constructed in Definition 3.1 for \( \bar{g}_n\alpha\beta \). Multiplication of \( g_n\alpha \) by \( \beta \) involves adding an edge \( e \) covering a bottom vertex \( b \) of \( g_n\alpha \). If \( e \) does not also cover a vertex \( w \) that is an essential cut vertex of \( g_n \) and that is at least \( n - 3 \) levels deep in \( g_n\alpha \) then \( P_n(\bar{T}_2) = P_n(\bar{T}_1) - 1 \).

We now use the above Lemmas, whose proofs are postponed to Section 4, to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( \alpha \in F \) be an element with \( l_n(\alpha) \leq 2n - 8 \). By Lemma 1, we may assume that the normal form of \( \alpha \) contains more negative than positive generators. Let \( k \) be the number of positive generators in the the normal form of \( \alpha \), and write \( \alpha = \alpha_+ \gamma \beta \), where \( \alpha_+ \) is the positive portion of the normal form of \( \alpha \), \( \gamma \) is the length \( k \) prefix of the negative portion of the normal form of \( \alpha \) and \( \beta \) is the remaining negative portion of the normal form. Write, \( \beta = \beta_1 \beta_2 \cdots \beta_m \), with \( \beta_i^{-1} \in X_n \). Thus, \( 2k + m = l_\infty(\alpha) \leq l_n(\alpha) \leq 2n - 8 \) and \( \frac{2k}{m} \leq n - 4 - k \).

By the construction of \( g_n \), multiplication of \( g_n \) by \( \alpha_+ \gamma \) involves changing only the essential portion of \( D(g_n) \). Let \( \bar{T} \) be the penalty tree for \( g_n\alpha_+\gamma \) given by Definition 3.1. By Lemma 3, \( P_n(\bar{T}) \leq P_n(g_n) \). Now, in the diagram for \( g_n\alpha_+\gamma \), no essential cut vertex of \( g_n \) is more than \( k \) levels deep. Thus, in the diagram
for \( g_n\alpha + \gamma \beta_1 \beta_2 \cdots \beta_i \), no essential cut vertex of \( g_n \) is more than \( k + i \) levels deep. Since \( \frac{m}{2} \leq n - 4 - k \), by Lemma 5 penalty tree \( \tilde{T}_i \) for \( g_n\alpha + \gamma \beta_1 \beta_2 \cdots \beta_i \) satisfies \( P_n(\tilde{T}_i) = P_n(\tilde{T}_{i-1}) - 1 \). Therefore,

\[
P_n(\tilde{T}(\frac{m}{2})) = P_n(\tilde{T}) - \left\lfloor \frac{m}{2} \right\rfloor \leq P_n(g_n) - \left\lfloor \frac{m}{2} \right\rfloor .
\]

Since \( l_\infty(g_n\alpha + \gamma \beta_1 \beta_2 \cdots \beta_i) = l_\infty(g_n) + \left\lfloor \frac{m}{2} \right\rfloor \), we therefore have,

\[
l_n(g_n\alpha + \gamma \beta_1 \beta_2 \cdots \beta_i) \leq l_n(g_n) - \left\lfloor \frac{m}{2} \right\rfloor .
\]

This implies that,

\[
l_n(g_n\alpha) = l_n(g_n\alpha + \gamma \beta) \leq l_n(g_n) - \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \leq l_n(g_n),
\]

proving that \( g_n \) is a dead end of depth at least \( 2n - 7 \) with respect to the generating set \( X_n \). Therefore, the dead end depth of \( F \) with respect to \( X_n \) is at least \( 2n - 7 \).

\[\square\]

4. Proofs

**Proof of Lemma 2.** Let \( w \) and \( b \) be as in the statement of the lemma. Let \( p \) be the path in \( \tilde{T} \) from \( b \) to \( w \). Let \( \alpha = \alpha_+\alpha_- \) be the normal form expression for \( \alpha \) with \( \alpha_+ \) consisting of the positive generator sequence and \( \alpha_- \) the negative generator sequence. We claim that every vertex of \( p \) is on the bottom path of \( g_n\alpha_+ \). This is certainly true for \( w \). Suppose towards a contradiction that there is a vertex in \( p \) not on the bottom path of \( g_n\alpha \). Let \( w' \) be the first such vertex along \( p \) going from \( w \) to \( b \). This means that \( w' \) is covered by a lower edge \( e \) of \( g_n\alpha_+ \). Now \( w' \) must have been reached by traveling along an edge in \( g_n\alpha = g_n\alpha_+\alpha_- \) reaching farthest to the left from its terminal vertex, which by assumption lies on the bottom path of \( g_n\alpha_+ \). This is impossible, because \( w' \) is covered by the edge \( e \) of \( g_n\alpha_+ \). This contradiction establishes the claim that every vertex in \( p \) is on the bottom path of \( g_n\alpha_+ \).

Now, if the active portion of the bottom path \( g_n \) contains \( N \) vertices, then \( g_n\alpha_+ \) contains \( N + k \) vertices in the active portion of its bottom path since \( \alpha_+ \) has the effect of removing \( k \) edges from \( g_n \). On the other hand, \( g_n\alpha \) contains \( N \) vertices on the active portion of its bottom path, so there are at most \( k \) vertices that are exposed in \( g_n\alpha_+ \) but not exposed in \( g_n\alpha \). Since no vertex along \( p \) except \( b \) is exposed in \( g_n\alpha \), the length of \( p \) is at most \( k \).

\[\square\]

**Proof of Lemma 3.** Since the tree \( \tilde{T} \) constructed in Definition 3.1 contains every penalty type vertex of \( g_n\alpha \), it is a penalty tree for \( g_n\alpha \). Working backwards through the construction of \( \tilde{T} \), we see that a vertex added in step 3 or 4 is a leaf of \( \tilde{T} \), so is not weighted. By Lemma 2, any vertex \( v \) added in step 2 is at most distance \( k \) through \( \tilde{T} \) to any essential cut vertex, \( w \) of \( g_n \) to its right. Such vertices \( w \) are the only possible leaves to the right of \( v \) through \( \tilde{T} \). Since \( 2k \leq 2n - 8 \) such vertices are at a distance less than \( n - 1 \) from leaves through \( \tilde{T} \) and therefore not weighted. Thus, the only weighted vertices of \( \tilde{T} \) in the active portion of \( g_n \) are those in the essential portion of the bottom path of \( g_n \). Since only the active portion of \( g_n \) is modified by multiplication by \( \alpha \), no vertex on the bottom path of \( g_n \) that is not weighted in \( T \) can become weighted in \( \tilde{T} \). Since the positive and negative portions
of $\alpha$ are the same length, $\alpha g_n$ and $g_n \alpha$ have the same number of vertices on the essential portion of their bottom paths. Thus, $P_n(T) \leq P_n(T_n) = P_n(g_n)$.

Proof of Lemma 4. Let $w, b$ and $j$ be as in the statement of the lemma. Every edge of the path $p$ in $\tilde{T}$ from $w$ to $b$ terminates either at a vertex on the bottom path of $g_n \alpha$ or a vertex that is the initial vertex of a lower edge terminating to the right of $w$ and thus covering $w$. Therefore, $d_{\tilde{T}}(w, b) \leq j$.

Proof of Lemma 5. Let $\alpha$ and $\beta$ be as in the statement of the lemma. By the construction of $g_n \alpha$ cannot expose any upper edge of $g_n$. Thus, $\beta$ indeed adds an edge $e$ to the diagram of $g_n \alpha$. Let $b$ be the bottom vertex of the diagram of $g_n \alpha$ that $e$ covers. Suppose that every essential cut vertex of $g_n$ that $e$ covers, it does so to a depth of less than $n - 3$. Let $e_1$ be the edge originating at $b$ and terminating at $t(e)$. By abuse of notation, we may think of $\tilde{T}_1$ as being a subtree of $D(g_n \alpha \beta)$. Penalty tree $\tilde{T}_2$ is related to $\tilde{T}_1$ by $\tilde{T}_2 = (\tilde{T}_1 \setminus \{e_1\}) \cup \{e\}$. Now, $b$ is not among the last $n - 2$ bottom vertices in the essential portion of $g_n \alpha$. Therefore, $b$ is a weighted penalty vertex in $\tilde{T}_1$. By Lemma 4, no essential cut vertex of $g_n$ covered by $e$ is at a distance of more than $n - 3$ from $b$ through $\tilde{T}_1$. Thus, in the tree $\tilde{T}_2$, $b$ is not a weighted vertex. Since the remaining weighted vertices of $\tilde{T}_2$ are the same as those of $\tilde{T}_1$, we have $P_n(\tilde{T}_2) = P_n(\tilde{T}_1)$.

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